nient reference book, however, a total rearrangement and revision would be necessary, as has been suggested by Ansari and Fatima in their detailed critical review in *Studies in History of Medicine* **8** (1984), 67–87.

**Über die Entstehung von David Hilberts "Grundlagen der Geometrie".** By M.-M. Toepell. Studien zur Wissenschafts-, Sozial-, und Bildungsgeschichte der Mathematik, Band 2. Göttingen (Vandenhoeck & Ruprecht). 1986. 293 + xiv pp. DM 78.

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Even in Hilbert's time, the appearance of his book [Hilbert 1899] on the foundations of geometry presented a paradox: a radical reformulation of material that was either elementary and well known or else new but so counterintuitive and extreme as to seem unlikely ever to achieve central importance. Today, as the teaching of Euclidean geometry and the content of the *Elements* recedes even further from the school curriculum, and the novel geometries Hilbert introduced are still little known, the paradox is if anything starker. What was such a great mathematician, then at the height of his powers, doing with such insubstantial material? The answer is not to be found in the social history of mathematics (a topic Toepell rightly does not even discuss): there was no clamor from the schools or the engineering faculties for geometers to put their house in order. But there was a crisis of sorts within geometry itself. The 19th century had seen a marked growth in the study of projective geometry, and people like Cayley and Klein felt that this geometry was more fundamental than Euclid's. There was also the discovery of non-Euclidean geometry to come to terms with. Unlike projective geometry, this geometry made physical sense; the two together made it clear that Euclid's seeming logicality was deeply flawed. The significance of Hilbert's work lies not so much in the rich details accompanying his analysis of geometry as in the profundity of his methodology. It is one of the merits of Toepell's book to present the details clearly and fully while never letting the reader lose sight of the importance of the novel methodology.

Traditionally, mathematicians asked: "what is mathematics about?" What, that is to say, are the objects with which it deals? In the case of geometry this approach is particularly natural, because it was argued that geometry deals with, refines, and organizes our intuitions about space. This was the view of Pasch [1882], in a book that greatly influenced Hilbert. Pasch said that his basic theorems were "grounded immediately in observation" (quoted in the book under review on p. 8). But Pasch then proceeded to develop vigorously a strictly logical series of deductions from his initial premises. It was this idea that caught Hilbert's imagination, and Hilbert soon turned his attention away from the objects of geometry and toward the study of the relations between the basic assumptions. This is one of the crucial starting places of the formal axiomatic method, and that method in the developed form of [Hilbert 1899] was what caused the excitement, well conveyed here (p. 257) by an interesting letter from Hurwitz to Hilbert written in 1903. Hurwitz wrote: "You have opened up an immeasurable field of mathematical investigation which can be called the 'mathematics of axioms' and which goes far beyond the domain of geometry."

Toepell shows that Hilbert's route to this breakthrough was a long one. The surprise of those in Göttingen at what they saw as a new departure by the master of algebraic number theory may have blinded historians to the lectures on geometry that Hilbert gave at Königsberg from 1888. Toepell documents richly and fully what the archives at Göttingen can tell us about Hilbert's research. At first Hilbert seems to have found it a rather unexciting matter of polishing up an edifice that had, after all, stood for 2000 years, and thus an enterprise deprived of the excitement of new problems. A lecture by H. Wiener in 1891 spurred him on to the realization not only of the general validity of the axiomatic method, but also of the possibility of detecting new theorems. A letter from F. Schur to Klein, later published in an extended form as [Schur 1898], recalled Hilbert to geometry, and at this point the intensive work began. The axioms that Hilbert considered were already grouped according to the intuitive property that each formalized, but independence often remained to be established. For the metrical geometries, it emerged that separating questions of completeness from the Archimedean property led to the discovery (by Max Dehn) of non-Archimedean geometries. The theorems of Pascal and Desargues posed stimulating challenges: which sets of axioms entailed them, and which did not? Although such geometries may never have claimed too much of anyone's attention, they were the perfect setting for the axiomatic method, for they could never have been discovered without that method.

The axiomatic method thus appeared as both a discovery method and as a rigorous way of arguing. But the study of axiom systems contains not only independence results but also consistency theorems. Consistency of an axiom system is guaranteed by providing it with a model, and discovery of a model requires the exercise of intuition. Thus Hilbert, although somewhat different from Klein in this respect, was not a mere "formalist"; Toepell quotes interestingly from Hilbert's later writings in this regard.

There is much else to say, however briefly, about this book. The choice of axioms and the role Hilbert allowed intuition here is well discussed, as is the separation of geometry from analysis, counter to Klein's immersion of the one in the other. One can ask about the relation between geometry and coordinates: Toepell finds (p. 250) that Hilbert always had the algebraization of geometry in mind. One might have wished for a treatment that brought a few more mathematicians out of the shadows, and for one historical work [Nagel 1939] to have been taken on board. Other authors, such as Freudenthal, have written about the impact of Hilbert's ideas, and Toepell alludes to them, thereby giving his own book a usefully tight focus. For the same reason, discussion of the Italian writers (Peano, Pieri, and Veronese) is slight, although Toepell does suggest (p. 57) that the language barrier may not have been as impenetrable as some have claimed. In short, Toepell has written well and clearly about a wealth of material never before published, and has provided us with extensive quotations from it, thus enabling us (as Toepell puts it on p. 265) "to see the master in his workshop." For that we are considerably in his debt.

## REFERENCES

Hilbert, D. 1899. *Grundlagen der Geometrie*. Leipzig: Teubner.

Nagel, E. 1939. The formation of modern conceptions of formal logic in the development of geometry. *Osiris* **7**, 142–223.

Pasch, M. 1882. *Vorlesungen über neuere Geometrie*. Leipzig: Teubner.

Schur, F. 1898. Ueber den Fundamentalsatz der Geometrie. *Mathematische Annalen* **51**, 401–409.

**The Śulbasūtras of Baudhāyana, Āpastamba, Kātyāyana and Mānava with Text, English Translation and Commentary.** By S. N. Sen and A. K. Bag. New Delhi (Indian National Science Academy). 1983. viii + 293 pp. Rs. 85, $30.

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This edition presents the Sanskrit texts (in transliteration) of the four major śulbasūtras or treatises on the geometry of the construction of Vedic altars, together with new and reasonably accurate translations and competent commentaries. It joins the ranks of a number of similar editions of the śulbasūtras that have been produced in India in recent years. These include the edition of the *Baudhāyana* by S. Prakash and R. S. Sharma (New Delhi, 1968; reprinted New Delhi, 1980); that of the *Kātyāyana* by S. D. Khadilkar (Poona, 1974); and that of the same four śulbasūtras as are contained in the volume under review, by S. Prakash and U. Jyotishmati (Allahabad, 1979). None of these editions is referred to at all by Sen and Bag. Perhaps they finished their work before some of these editions appeared, and also before some other important contributions to the study of the śulbasūtras that they have ignored, such as those by R. P. Kulkarni (*Geometry According to Śulba Sūtra* [Poona, 1983]), A. Michaels (*Beweisverfahren in der vedischen Sakralgeometrie* [Wiesbaden, 1978] and *A Comprehensive Śulvasūtra Word Index* [Wiesbaden, 1983]), and T. A. Sarasvati (*Geometry in Ancient and Medieval India* [Delhi, 1979], pp. 14–60).

Had they been able to consult some of these books, the authors might not only have improved their texts and their understanding of them in several places, but